A transformation for non-homentropic flows, with an application to large-amplitude motion in the atmosphere

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A transformation has been found which reduces steady non-homentropic flows of a compressible fluid to homentropic flows, provided diffusive and gravity effects are negligible. With this transformation, the equation governing steady two-dimensional flows of a compressible fluid with variable entropy in a gravitational field is derived, which is then applied to the study of atmospheric waves in the lee of mountains. The corresponding equation governing swirling axisymmetric flows is also given.

1. Introduction

It is well known that the flow of gas behind a curved shock is non-homentropic. In the atmosphere, waves are sometimes formed in the lee of mountains as a result of the non-homentropy of air and of the action of gravity. But the equations governing non-homentropic flows are so complicated that no non-trivial exact solution exists. Therefore in existing literature on the formation of lee waves, the theory has always been based on perturbation methods, and is thus only valid if the amplitude of wave motion (i.e. the vertical displacement) is small.

In this paper a transformation will be presented which will render unnecessary the consideration of non-homentropy in steady flows of ideal gases, provided the effects of viscosity and of gravity are neglected, in the sense that every steady non-homentropic flow can be reduced thereby to a steady homentropic flow. Gravity effects can certainly be neglected in the aerodynamics of aircraft or flying objects, though it is of primary importance in the study of waves in the atmosphere. Viscous effects can, as usual, be neglected outside of the boundary layer, and are certainly of secondary importance in atmospheric flows. Thus the transformation to be presented here is not without practical value.

In the study of lee waves in the atmosphere, the presence of gravity, which is now of paramount importance, prevents the above-mentioned transformation from absorbing the effects of non-homentropy altogether. Nevertheless, the use of this transformation leads to the derivation of a much simplified equation for steady two-dimensional flows, which, for an atmosphere slightly stratified in entropy and in specific energy, possesses four essentially different classes of

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solutions, provided the Mach number is everywhere small, so that the effect of *dynamic* compressibility on density variation can be neglected. (In the first 10 or 15 km of the atmosphere, where lee waves can conceivably be expected to occur, the Mach number of atmospheric flows is certainly everywhere small.) The solutions are exact in the sense that the displacements are not assumed to be small, but can have any arbitrary values of the same magnitude as the scale of the motion.

An equation governing steady axisymmetric flows with swirl has also been obtained, which may be helpful for the study of the Hirsch-tube phenomenon and of tornadoes.

2. The transformation

To bring out the full significance for ordinary aerodynamics of the transformation mentioned in the Introduction, body forces will be ignored for the time being. The equations of motion for steady flows are then

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \tag{1}$$

in which u_i is the velocity component in the direction of the Cartesian coordinate x_i ($i = 1, 2, 3$), ρ is the density, p is the pressure, and the summation convention has been adopted. The equation of continuity is, exactly,

$$\frac{\partial u_j}{\partial x_j} + \frac{1}{\rho} u_j \frac{\partial \rho}{\partial x_j} = 0. \tag{2}$$

If the flow is homentropic, and the fluid is a perfect gas,

$$p/\rho^\gamma = \text{constant}, \tag{3}$$

in which γ is the ratio of the specific heat* at constant pressure (c_p) to that at constant volume (c_v), and the flow is governed by equations (1) to (3). If the flow is not homentropic but diffusion is neglected, the entropy along a path line is constant. For steady flows this means that

$$u_j \frac{\partial}{\partial x_j}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{4}$$

which replaces equation (3), and states the constancy of entropy along a streamline in steady flows.

One of the great difficulties encountered in dealing with non-homentropic flows is that, since the gas is no longer barotropic (which is another way of saying that a single relationship between p and ρ does not exist), ρ cannot be absorbed into the pressure term in (1), and the motion, even started from rest, will not continue to be irrotational. The system consisting of equations (1), (2), and (4) is so complicated that at present no non-trivial solutions exist. However, the situation is not as bad as it appears at first sight, as the transformation now to be given will show.

* Both c_p and c_v are assumed to be constant.

In virtue of (4),

$$u_j \frac{\partial F(\lambda)}{\partial x_j} = 0, \tag{5}$$

in which $F(\lambda)$ is any arbitrary function of λ, defined by

$$\lambda = \frac{\rho}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma}, \tag{6}$$

with p_0 denoting a reference pressure and ρ_0 a reference density. Since the entropy S is connected with p, ρ, and c_v by

$$p/\rho^\gamma = \text{constant} \times e^{S/c_v}, \tag{7}$$

the quantity λ is connected with the entropy by

$$\lambda = \text{constant} \times e^{-S/c_p}, \tag{8}$$

and is to be determined *for each streamline* from the upstream conditions, by virtue of its constancy along a streamline in steady flows, as stated in (5). With the transformation

$$u_i' = \sqrt{\lambda}\, u_i, \quad \rho' = \rho/\lambda, \quad \text{and} \quad p' = p, \tag{9}$$

equations (1) and (2) become, in virtue of (5),

$$u_j' \frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho'}\frac{\partial p'}{\partial x_i} \tag{10}$$

and

$$\frac{\partial u_j'}{\partial x_j} + \frac{1}{\rho'} u_j' \frac{\partial \rho'}{\partial x_j} = 0. \tag{11}$$

Furthermore, the last two of equations (9) and equation (6) can be combined to give

$$\frac{p'}{\rho'^\gamma} = \frac{p}{\rho'^\gamma} = \frac{p_0}{\rho_0^\gamma} = \text{constant}. \tag{12}$$

Now equations (10), (11), and (12) are identical in form to (1), (2), and (3), and hence govern homentropic flows in terms of the primed quantities. But this means that to any solution of equations (10) to (12) representing a homentropic flow in terms of u_i', ρ', and p', there corresponds a non-homentropic flow in terms of u_i, ρ, and p, which are obtained from equations (9), and vice versa. Consideration of boundary conditions does not affect this conclusion. For convenience the flow in terms of the primed quantities will be called the associated flow.

The *associated* flow may not be irrotational. But if it originates from a big reservoir, where the fluid is at rest and therefore possesses no vorticity, irrotationality (in the u_i'-field) will persist downstream. This can be seen by eliminating p' from (10) by cross-differentiation, producing

$$u_j' \frac{\partial \xi_i'}{\partial x_j} = \xi_j' \frac{\partial u_i'}{\partial x_j} + \frac{\xi_i'}{\rho'} u_j' \frac{\partial \rho'}{\partial x_j}. \tag{13}$$

in which ξ_i' is the ith component of the vorticity in the u_i'-field. Equation (13) clearly indicates the persistence of irrotationality (or of the vanishing of ξ_i').

If $\xi'_i = 0$, integration of equations (10) in the usual way produces (with q' as the speed in the associated flow)

$$\tfrac{1}{2}q'^2 + \frac{\gamma}{\gamma-1}\frac{p'}{\rho'} = \tfrac{1}{2}q'^2_{\max} = \text{constant}, \tag{14}$$

which is valid for the whole field of flow. This equation can also be written

$$\tfrac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{1}{2\lambda}q'^2_{\max} = \text{constant} \times \frac{p^{1/\gamma}}{\rho}, \tag{15}$$

in which q is the speed of the actual flow. If $\xi'_i \neq 0$,

$$\tfrac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = f(S), \tag{16}$$

in which $f(S)$ is an arbitrary function of the entropy S.

Equation (14) is simply the ordinary Bernoulli equation. In fact, if gravity is not neglected the complete equation is (with z measured vertically upward)

$$\tfrac{1}{2}q^2 + gz + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = f(S), \tag{17}$$

which is well known.

Since the transformation presented in this section may suggest a similarity in its underlying idea with Crocco's stream function, it is desirable to point out the essential differences between the two inventions. Crocco dealt with the homenergic (constant specific energy, or constant $f(S)$ in (17)) but non-homentropic flows behind a shock, and, utilizing the constancy of entropy along a streamline in steady flows, obtained the equation of continuity for two-dimensional flow* (in the usual notation)

$$\frac{\partial}{\partial x}[u(q^2_{\max}-q^2)^{1/(\gamma-1)}] + \frac{\partial}{\partial y}[v(q^2_{\max}-q^2)^{1/(\gamma-1)}] = 0, \tag{18}$$

which permits the use of a stream function ψ (Crocco's stream function), in terms of which the velocity components can be expressed

$$u = (q^2_{\max}-q^2)^{-1/(\gamma-1)}\frac{\partial\psi}{\partial y}, \quad v = -(q^2_{\max}-q^2)^{-1/(\gamma-1)}\frac{\partial\psi}{\partial x}. \tag{19}$$

Close examination of the factor

$$(q^2_{\max}-q^2)^{1/(\gamma-1)}$$

shows that it is really just the density ρ multiplied by a function of the entropy. To this extent there is some similarity between the present transformation and Crocco's invention. But the similarity stops here. The important differences are: (*a*) Crocco's development is only for homenergic flows, whereas the present transformation deals with non-homenergic and non-homentropic flows; (*b*) Crocco's development is only for two-dimensional or axisymmetric flows, whereas the present transformation deals with general three-dimensional flows; (*c*) Crocco invented a new stream function but left the velocity unchanged, hence did not arrive at the transformation embodied in (9).

* The development for axisymmetric flows is similar.

We shall now take stock and see what conclusions can be drawn from the transformation embodied in (9). As is evident, the associated flow has the same *pattern* as the actual flow. It (the associated flow) is irrotational if the actual flow originates from a large reservoir where the gas is at rest, or, more generally, if the associated flow is irrotational far upstream. Whether the associated flow is irrotational or not, the third of equations (9) ensures that the drag and lift on any body placed in the gas stream will be the same as that calculated from the associated flow. If far upstream the actual flow* is unidirectional, with constant velocity but variable entropy, the associated flow will be rotational, and there will in general be lift on a body placed in the stream. This is an example illustrating how entropy stratification upstream can give rise to lift on a body moving with constant velocity in a quiescent but stratified gas.

The above conclusions, and indeed the transformation embodied in (9), are based on the conservation of entropy along each streamline. Therefore flows with shocks must be considered anew, if they are to be considered *as a whole* and not as a collection of separate regions. Looking at such flows in their entirety, we can draw some interesting conclusions in spite of the entropy change across the shock. Since three-dimensional shocks differ from two-dimensional ones only in complexity, not in principle, only two-dimensional shocks will be considered here. The pre-shock flow is assumed to be parallel to the x-axis, with velocity u_1 (which may vary with y), with the subscript 1 now referring to pre-shock flow and 2 to post-shock flow. The shock wave is in general curved and the post-shock flow in general non-parallel. With u and v denoting velocity components in the x- and y-directions, and β denoting the local angle of inclination of the shock wave, continuity demands (Liepmann & Puckett 1947, p. 51) that along the shock wave

$$\rho_1 u_1 \sin\beta = \rho_2(u_2 \sin\beta - v_2\cos\beta), \tag{20}$$

in which ρ is the density. The conservation of momentum normal to the shock wave demands that, along the shock wave,

$$p_1 + \rho_1 u_1^2 \sin^2\beta = p_2 + \rho_2(u_2\sin\beta - v_2\cos\beta)^2, \tag{21}$$

in which p is the pressure. The conservation of momentum parallel to the shock wave demands

$$p_1 u_1^2 \sin\beta\cos\beta = \rho_2(u_2\sin\beta - v_2\cos\beta)(u_2\cos\beta + v_2\sin\beta), \tag{22}$$

and the energy equation remains

$$\tfrac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \tfrac{1}{2}(u_2^2+v_2^2) + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}. \tag{23}$$

Equations (20) to (23) contain five unknowns: u_2, v_2, ρ_2, p_2, and β. The variation of β from place to place along the shock wave can only be determined by the equations governing the flow before the shock, those governing post-shock flow, equations (20) to (23), and the conditions at the solid boundaries and at infinity, by a trial-and-error process.

* The flow is the steady flow equivalent to the flow caused by a body moving with constant velocity in a quiescent but stratified gas.

Whereas the transformation embodied in (9) does not obviate this tedious process, it does throw some light on shock waves ahead of which the fluid is already non-homentropic. With λ defined by (6) and determined for each streamline (even after it pierces through the shock wave) by the upstream or pre-shock condition, we insist on making the transformation represented by (9), in spite of the abrupt increase of entropy along each streamline as it crosses the shock wave. The associated flow is parallel and homentropic upstream, and is irrotational if λu_1^2 is constant throughout. Behind the shock even the associated flow is not homentropic, but is governed by the equations

$$\left(u'\frac{\partial}{\partial x}+v'\frac{\partial}{\partial y}\right)(u',v') = -\frac{1}{\rho'}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)p',$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{1}{\rho'}\left(u'\frac{\partial}{\partial x}+v'\frac{\partial}{\partial y}\right)\rho' = 0,$$

$$\left(u'\frac{\partial}{\partial x}+v'\frac{\partial}{\partial y}\right)\left(\frac{p'}{\rho'^{\gamma}}\right) = 0,$$

because λ is constant on each streamline by imposition. These equations, with the primes dropped, are identical with the equations governing post-shock flow. The transformation (9) achieves homentropy in the pre-shock region, but otherwise leaves the governing equations unchanged in form. At the shock wave, (20) to (23) are still valid if all the quantities (except, of course, β and γ) are primed. Thus, even when shock waves are present, to every flow with entropy stratification before the shock corresponds an associated flow of the *same pattern for the entire field of flow*, with homentropy (through not necessarily irrotationality) before the shock. The actual lift and drag on a body placed in the gas stream are the same as those calculated from the associated flow.

Although gravity has been neglected in this section, equations (9) are still helpful when gravity is taken into account, because they simplify the governing equations a great deal, as will be shown in the following sections.

3. Equation governing two-dimensional flows in a gravitational field

If (x_1, x_3) and (u_1, u_3) are now written as (x, z) and (u, w), the equations of motion for steady two-dimensional flows are, with the gravity included,

$$\rho\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x},$$

$$\rho\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}-g\rho,$$

in which z is measured in a direction opposite to that of the gravitational acceleration g. With the transformation embodied in (7), the equations of motion become

$$u'\frac{\partial u'}{\partial x}+w'\frac{\partial u'}{\partial z} = -\frac{1}{\rho'}\frac{\partial p'}{\partial x}, \tag{24}$$

$$u'\frac{\partial w'}{\partial x}+w'\frac{\partial w'}{\partial z} = -\frac{1}{\rho'}\frac{\partial p'}{\partial z}-g\lambda. \tag{25}$$

The equation of continuity $$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho w)}{\partial z}=0$$

now has the form $$\frac{\partial(\rho' u')}{\partial x}+\frac{\partial(\rho' w')}{\partial z}=0,$$

which permits the use of the stream function ψ' such that

$$u'=\frac{1}{\rho'}\frac{\partial\psi'}{\partial z},\quad w'=-\frac{1}{\rho'}\frac{\partial\psi'}{\partial x}.$$

With $$q'^2=u'^2+w'^2,\quad I'=\frac{q'^2}{2}+\int\frac{dp'}{\rho'},$$

$$H'=\frac{q'^2}{2}+gz\lambda+\int\frac{dp'}{\rho'}\quad\text{and}\quad \eta'=\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x},\tag{26}$$

the equations of motion can be written

$$\frac{\eta'}{\rho'}\frac{\partial\psi'}{\partial x}=\frac{\partial I'}{\partial x},\tag{27}$$

$$\frac{\eta'}{\rho'}\frac{\partial\psi'}{\partial z}=\frac{\partial I'}{\partial z}+g\lambda.\tag{28}$$

Multiplication of (27) by dx and (28) by dz and addition of the resulting equations produces

$$\frac{\eta'}{\rho'}d\psi'=dI'+g\lambda\,dz=dH'-gz\,d\lambda.$$

But since H and λ are functions of ψ' alone,

$$\frac{\eta'}{\rho'}+gz\frac{\partial\lambda}{\partial\psi'}=\frac{dH'}{d\psi'}=h(\psi'),\tag{29}$$

or, from equation (25) and the last of equations (26),

$$V^2\psi'-\frac{1}{\rho'}\left(\frac{\partial\rho'}{\partial x}\frac{\partial\psi'}{\partial x}+\frac{\partial\rho'}{\partial z}\frac{\partial\psi'}{\partial z}\right)+gz\rho'^2\frac{d\lambda}{d\psi'}=\rho'^2h(\psi'),\tag{30}$$

which is the desired equation. The density ρ' has to be evaluated from the third of equations (26), in which $H'(\psi')$ and $\lambda(\psi')$ are determined from upstream conditions.

4. Swirling axisymmetric flows in a gravitational field

Cylindrical co-ordinates (r,θ,z) will be used. In these co-ordinates, the velocity components will be denoted by u, v, and w, respectively. The equations of motion for steady axisymmetric flows are, with viscosity neglected,

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r},\tag{31}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=0,\tag{32}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g,\tag{33}$$

in which z is again measured in the direction opposite to that of the gravitational acceleration. The equation of continuity is

$$\frac{\partial(r\rho u)}{\partial r}+\frac{\partial(r\rho w)}{\partial z}=0.$$

By the transformation indicated by equations (7), equations (31) to (33) become

$$u'\frac{\partial u'}{\partial r}+w'\frac{\partial u'}{\partial z}-\frac{v'^2}{r}=-\frac{1}{\rho'}\frac{\partial p'}{\partial r}, \tag{34}$$

$$u'\frac{\partial v'}{\partial r}+w'\frac{\partial v'}{\partial z}+\frac{u'v'}{r}=0, \tag{35}$$

$$u'\frac{\partial w'}{\partial r}+w'\frac{\partial w'}{\partial z}=-\frac{1}{\rho'}\frac{\partial p'}{\partial z}-g\lambda, \tag{36}$$

and the equation of continuity becomes

$$\frac{\partial(r\rho' u')}{\partial r}+\frac{\partial(r\rho' w')}{\partial z}=0. \tag{37}$$

Equation (37) permits the use of a stream function ψ' in terms of which the velocity components can be expressed:

$$u'=-\frac{1}{r\rho'}\frac{\partial\psi'}{\partial z},\quad w'=\frac{1}{r\rho'}\frac{\partial\psi'}{\partial r}. \tag{38}$$

Equation (35) expresses the conservation of angular momentum for the same particle, because it can be written as

$$u'\frac{\partial(rv')}{\partial r}+w'\frac{\partial(rv')}{\partial z}=0.$$

Consequently rv' is a function of ψ' alone. For convenience, we take

$$(rv')^2=f(\psi'). \tag{39}$$

With $\quad q'^2=u'^2+v'^2+w'^2,\quad J'=\dfrac{u'^2+w'^2}{2}+\displaystyle\int\frac{dp'}{\rho'},\quad I'=\dfrac{q'^2}{2}+\displaystyle\int\frac{dp'}{\rho'},$

$$H'=\frac{q'^2}{2}+gz\lambda+\int\frac{dp'}{\rho'},\quad \eta'=\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial r}, \tag{40}$$

equations (34) and (36) can be written, with the aid of (38) and (39)

$$\frac{\eta'}{r\rho'}\frac{\partial\psi'}{\partial r}-\frac{f(\psi')}{r^3}=-\frac{\partial J'}{\partial r},$$

$$\frac{\eta'}{r\rho'}\frac{\partial\psi'}{\partial z}=-\frac{\partial J'}{\partial z}-g\lambda,$$

or

$$\frac{\eta'}{r\rho'}\frac{\partial\psi'}{\partial r}-\frac{1}{2r^2}\frac{\partial}{\partial r}f(\psi')=-\frac{\partial I'}{\partial r}, \tag{41}$$

$$\frac{\eta'}{r\rho'}\frac{\partial\psi'}{\partial z}-\frac{1}{2r^2}\frac{\partial}{\partial z}f(\psi')=-\frac{\partial I'}{\partial z}-g\lambda. \tag{42}$$

Multiplication of (41) by $-dr$ and (42) by $-dz$ and addition of the results gives

$$-\frac{\eta'}{r\rho'}+\frac{1}{2r^2}\frac{df}{d\psi'}=\frac{dH'}{d\psi'}-gz\frac{d\lambda}{d\psi'}, \tag{43}$$

or, with the aid of (38) and the last of equations (40),

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\psi'-\frac{1}{\rho'}\left(\frac{\partial\rho'}{\partial r}\frac{\partial\psi'}{\partial r}+\frac{\partial\rho'}{\partial z}\frac{\partial\psi'}{\partial z}\right)+\frac{\rho'^2}{2}\frac{df}{d\psi'}+g\rho'^2zr^2\frac{d\lambda}{d\psi'}=r^2\rho'^2\frac{dH'}{d\psi'}, \tag{44}$$

which is the equation governing swirling axisymmetric motion in a gravitational field. In many engineering applications (such as to the Hirsch tube), the term involving g can be neglected. Again, ρ' is to be calculated from the equation involving H' in (40), in which the functions $H'(\psi')$ and $\lambda(\psi')$ are again to be determined from upstream conditions. Equation (44) could serve as a starting point for the study of tornadoes.

5. Lee waves of large amplitude

The phenomenon of gravity waves in air (considered as a compressible fluid) in the lee of mountain ridges has been studied by Lyra, Queney, Corby, Scorer, and others, and recently by Crapper (1949). (For references, see Crapper's work.) Perturbation methods have been used by all of these authors, so that their results do not apply to large vertical displacements. Batchelor (1953) gave an equation governing homentropic flows in the atmosphere, obtained on the assumption of irrotationality (which is a consequence of homentropy if the motion has been started from rest) and small Mach number in the entire flow field. The effect of gravity is retained in Batchelor's equation, so that it applies to large vertical displacements under the assumptions stated. However, since wave motions in the atmosphere are essentially due to non-homentropy or non-homenergy, Batchelor's equation cannot be applied to a study of lee waves.

The exact equation governing steady two-dimensional non-homentropic flows is equation (30), which can be used to study lee waves. However, the equation is so very complicated that no solution can be obtained without some simplifying assumptions. With Batchelor, we shall assume that the Mach number is everywhere small, so that, *a fortiori*, the *variation* of the square of the speed is small compared with the square of the sound speed. Therefore the density variation due to the variation of speed is negligible, and any change in the density along the same streamline is due to change of elevation alone. Since isentropy along a streamline is the most important assumption underlying the derivation of (29), it might appear that the simplifying assumption on ρ (hence on ρ') implies that the pressure is a function of elevation alone for any streamline. Such an implication must not be inferred from the assumption on the density, because the pressure must be calculated from the equation containing H' in (26), with the term $\frac{1}{2}q'^2$ included. Indeed, if the pressure were only dependent on z on any one streamline, scarcely any non-trivial motion would be possible. The situation is not unlike that encountered in the study of free-convection problems, in which the fluid is assumed incompressible as far as continuity and the inertia of the fluid is concerned, but is considered to have a variable density as far as the important term representing body force is concerned. Another similar situation is encountered in the study of incompressible fluids. If entropy is assumed constant along a path line, surely there is some relation connecting the density to the pressure on such a line. But the assumption of constant density does not imply constant pressure, because the pressure can change a great deal for an infinitesimal change in the density of what is normally considered to be an incompressible fluid. In the present case, the assumption concerning ρ' affects only the first term in (29) or the first two groups of terms in (30), and is an assumption concerning essentially the continuity equation only. The inertia effect of density change has been absorbed once and for all in the transformation represented by equation (9). Gravity force is exactly represented by $gz(d\lambda/d\psi')$ and the force resulting from pressure gradient is represented by $dH'/d\psi'$ in (29), with H' given by the *full* expression in (26). The factor ρ'^2 in the last two terms of (30) appears from a common multiplication by that factor, and does not affect the physical reasoning given above.

If the variation of H' or of λ with ψ' is large, equation (30) is still too difficult to solve. Therefore we shall assume the variation of H' and λ to be small. As far as the calculation of ρ' is concerned, we shall ignore the variation of H' and λ altogether and justify the procedure by the same arguments as those presented in the last paragraph. For ρ', then, calculation from the third of (26), with the term $\frac{1}{2}q'^2$ neglected and λ equal to 1 (if the reference density and pressure are those at some point in the atmosphere under discussion), yields

$$\frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^\gamma}\rho'^{\gamma-1} = H'-gz, \tag{45}$$

or

$$\rho'^{\gamma-1} = \frac{H'-gz}{K}, \quad K = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^\gamma}. \tag{46}$$

With (46), equation (30) becomes

$$\nabla^2\psi' + \frac{1}{\gamma-1}\frac{g}{H'-gz}\frac{\partial\psi'}{\partial z} + gz\left(\frac{H'-gz}{K}\right)^{2/(\gamma-1)}\frac{d\lambda}{d\psi'} = \left(\frac{H'-gz}{K}\right)^{2/(\gamma-1)} h(\psi'), \tag{47}$$

in which, as in (45) and (46), H' is considered to be a constant except in connexion with $h(\psi')$. It must be remembered that whatever assumptions have been made on ρ', H', and λ, they do not limit the amplitude of the vertical displacement of the motion in any way, or the slope of the streamlines. Freedom from such limitations is the chief merit of the present theory. The functions $\lambda(\psi')$ and $h(\psi')$ are to be determined from upstream conditions.

Equation (47) is exactly linear if

$$\frac{d\lambda}{d\psi'} = a\psi' + b, \quad h(\psi') = m\psi' + n.$$

Since ψ' can be changed by a constant, there are seven different cases:

(1) $a \neq 0, b = 0, m = n = 0$,
(2) $a \neq 0, b = 0, m = 0, n \neq 0$,
(3) $a \neq 0, b = 0, m \neq 0, n = 0$,
(4) $a \neq 0, b = 0, m \neq 0, n \neq 0$,
(5) $a = 0, b \neq 0, m = n = 0$,
(6) $a = 0, b \neq 0, m = 0, n \neq 0$,
(7) $a = 0, b \neq 0, m \neq 0, n = 0$.

We have not included the cases in which $a = b = 0$, because they correspond to homentropic flows. If in addition $m = n = 0$, the flow is in fact irrotational. But if $m \neq 0$, wave motion is possible. This wave motion is not due to non-homentropy because the entropy is constant for $a = b = 0$, but is due to non-homenergy.

Since z can be changed by a constant, the seven cases can be reduced to four essentially different cases. Thus cases (3), (4), and (6) are not essentially different from cases (1), (2), and (5), respectively. The four essentially different cases, are therefore, (1), (2), (5), and (7).

If

$$\Psi = \psi'/\psi_0, \quad \xi = x/d, \quad \eta = z/d, \tag{48}$$

in which d is a reference length which can be taken either to be the depth of the troposphere or the depth below some very stable layer, and ψ_0 is a reference stream function, the linear cases are represented by

$$\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)\Psi+\frac{1}{\gamma-1}\frac{1}{\alpha-\eta}\frac{\partial \Psi}{\partial \eta}+\eta(\alpha-\eta)^{2/(\gamma-1)}(A\Psi+B) = (\alpha-\eta)^{2/(\gamma-1)}(C\Psi+D), \tag{49}$$

in which

$$\alpha = H'/gd \tag{50}$$

is the ratio of the *equivalent* depth of the atmosphere, assumed completely homentropic (for defining this depth only), to the reference depth d. The value of α may vary over a range, but if d is taken to be 10 km (average depth for the troposphere) a representative value for α is 3·5.

In all the linear cases the solution can be put in the form

$$\Psi = \Psi_1(\eta)+\Psi_2(\xi,\eta), \tag{51}$$

in which both parts on the right-hand side satisfy (49). If we suppose that from $\eta = 1$ (or $z = d$) upwards the atmosphere is much more stable than the layer below, so that the vertical displacement at $\eta = 1$ is small compared with that prevailing in the layer $0 \leqslant \eta < 1$ (see Yih 1960*a*), a rigid plane may be imagined to be situated at $\eta = 1$. The boundary conditions for Ψ are then

$$\Psi_1(0) = 0, \quad \Psi_1(1) = E \text{ (a constant).} \tag{52}$$

Solution of the differential equation (49), with Ψ_1 replacing Ψ therein, together with (52) then yield a $\Psi_1(\eta)$ corresponding to an upstream condition which makes (47) linear. Although the actual upstream condition may not give rise to the linearity of (47), suitable choices of the constants A, B, C, D, and E can produce infinitely many upstream conditions one of which may approximate the actual upstream condition rather closely, while the linearity of (47) is maintained throughout. The method of approach is therefore an inverse one, as is often the case in classical aerodynamics.

As to Ψ_2, the boundary conditions are, with the gas flowing from $\xi = -\infty$ to $\xi = +\infty$,

$$\Psi_2 \to 0 \quad \text{as} \quad \xi = -\infty, \tag{53a}$$

$$\Psi_2(\xi, 1) = 0, \tag{53b}$$

$$\Psi = \Psi_1+\Psi_2 = 0 \quad \text{on the lower boundary (ground).} \tag{53c}$$

No boundary condition is imposed at $\xi = +\infty$, because waves may exist in the lee of a barrier, and when they do not exist the condition far downstream is automatically the same as that far upstream. The condition (53*a*) is imposed on the assumption that the existence of a barrier, the effect of which is represented by Ψ_2, does not influence the condition far upstream. The (assumed) rigidity of the boundary at $\eta = 1$ demands (53*b*). The satisfaction of (53*c*) on the surface of a ground of given profile is difficult. But the inverse method given in another paper (Yih 1960*b*) can again be used. As described in that paper, the auxillary condition

$$\Psi_2(\xi, 0) = 0 \tag{53d}$$

is imposed to guarantee good behaviour of the solution far downstream, where the ground is assumed level and where the lee-wave components contained in Ψ_2 do not die out exponentially. The streamline $\Psi = 0$ therefore consists of two branches: the line $\eta = 0$ and the surface of the ground. As in the case of lee-wave formation in an incompressible fluid (Yih 1960*b*), the amplitudes of the lee-wave components depend only on certain integral properties of the barrier, and not on its detailed shape.

Although the calculation for the details of lee-waves governed by (49) involves the detailed calculations for the eigenfunctions, in case (7) at least, the number of lee-wave components can be predicted from the value of C without detailed calculations. Thus, the function Ψ_1 satisfies the equation (with $A = D = 0$ in case (7))

$$\Psi_1'' + \frac{1}{\gamma-1}\frac{1}{\alpha-\eta}\Psi_1' + \eta(\alpha-\eta)^{2/(\gamma-1)}B = (\alpha-\eta)^{2/(\gamma-1)}C\Psi_1,$$

in which the primes indicate ordinary differentiation. The other part of the solution consists of terms of the form

$$\Psi_2 = \begin{Bmatrix}\sin k\xi\\ \cos k\xi\end{Bmatrix} f(\eta),$$

in which k may be imaginary, and f satisfies the equation

$$f'' + \frac{1}{\gamma-1}\frac{1}{\alpha-\eta}f' - [C(\alpha-\eta)^{2/(\gamma-1)} + k^2]f = 0. \tag{54}$$

With

$$\zeta = \frac{\gamma-1}{\gamma}(\alpha-\eta)^{\gamma/(\gamma-1)}, \tag{55}$$

equation (54) becomes

$$\frac{d^2f}{d\zeta^2} - \left[C + k^2\left(\frac{\gamma}{\gamma-1}\zeta\right)^{-2/\gamma}\right]f = 0. \tag{56}$$

The interval $0 \leqslant \eta \leqslant 1$ is now transformed to

$$\frac{\gamma-1}{\gamma}(\alpha-1)^{\gamma/(\gamma-1)} \leqslant \zeta \leqslant \frac{\gamma-1}{\gamma}\alpha^{\gamma/(\gamma-1)}.$$

Let

$$l = \frac{\gamma-1}{\gamma}[\alpha^{\gamma/(\gamma-1)} - (\alpha-1)^{\gamma/(\gamma-1)}]. \tag{57}$$

By equating k to zero and computing the values for C which will enable f to satisfy the boundary conditions corresponding to (53*b*) and (53*d*), i.e.

$$f(0) = 0, \quad f(1) = 0,$$

we have $$C_n = -(n\pi/l)^2.$$

If $$(n\pi/l)^2 \leqslant -C < \{[(n+1)\pi]/l\}^2,$$

there are n non-negative eigenvalues for k^2, and hence n lee-wave components.

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REFERENCES

BATCHELOR, G. K. 1953 The conditions for dynamic similarity of motions of a frictionless perfect-gas atmosphere. *Quant. J. Roy. Soc. Met.* **79**, 224–35.

CRAPPER, G. D. 1959 A three-dimensional solution for waves in the lee of mountains. *J. Fluid Mech.* **6**, 51–76.

LEIPMANN, H. W. & PUCKETT, A. E. 1947 *Introduction to Aerodynamics of a Compressible Fluid.* New York: Wiley and Sons.

YIH, C.-S. 1960*a* Gravity waves in a stratified fluid. *J. Fluid Mech.* **8**, 481–508.

YIH, C.-S. 1960*b* Exact solutions for steady two-dimensional flows of a stratified fluid. *J. Fluid Mech.* **9**, 161–174.